ELSEVIER
Topology and its Applications 95 (1999) 229–250
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Coincidence theory on the complement

Jianhan Guo, Philip R. Heath *

Department of Mathematics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada A1C 5S7

Received 17 May 1997; received in revised form 8 December 1997

Abstract

In this work we generalize two aspects of Nielsen fixed point theory on the complement to Nielsen coincidence theory. The first aspect concerns the location (under relative homotopies) of coincidence points. It prepares the way for equivariant coincidence theory and the for second part. A minimum theorem is forthcoming under the condition that the subspace can be by-passed. The second aspect (the study of surplus periodic points on the complement) gives a parallel (but quite different) theory when the subspace cannot be by-passed.

Other features of this work include a modified fundamental group approach which simplifies the exposition. Secondly in addition to the usual Jiang condition it includes an analogue of it which ensures that the Reidemeister and Nielsen numbers are the same when the Lefschetz number is nonzero.

Keywords: Nielsen coincidence numbers; Relative Nielsen numbers; Coincidence points on the complement; Location of coincidence points; Jiang space

AMS classification: 55M20

1. Introduction

In this paper we generalize two aspects of relative fixed point theory to give the corresponding concepts for relative Nielsen coincidence theory. Both aspects are clearly of interest in their own right, but the first is also important in the development of equivariant coincidence theory ([5,7] and [8]—see also Example 3.18 and the remarks following it).

Let $f, g : X \to Y$ be maps, a coincidence of f and g is a point $x \in X$ such that $f(x) = g(x)$. Clearly coincidence theory generalizes fixed point theory for the case that $X = Y$, and g is the identity on X. If $f, g : X \to Y$ are maps, where X and Y are closed oriented manifolds of the same dimension, then a Nielsen number $N(f, g)$ has been defined (i.e.,

* Corresponding author. E-mail: pheath@math.mun.ca.

0166-8641/99/$ – see front matter
PII: S0166-8641(98)00006-6

[15,1]), which is a lower bound for the number of coincidences of maps that are homotopic to f and g. This bound is known to be sharp for certain classes of manifolds. As with fixed point theory, if we put constraints on the maps involved then this can significantly alter the lower bound. For example, if we consider relative maps $f, g : (X, A) \to (Y, B)$, then the minimum number of coincidences when we restrict to relative homotopies of f and g may be significantly larger than $N(f, g)$. A relative coincidence number has been introduced to deal with this situation in [13] and [12] (it is denoted by $N_{rel}(f, g)$ in these references, but by $N(f, g; X, A)$ here). In this work we generalize to coincidence theory two variations of relative fixed point theory due to Zhao ([18] and [19]). We illustrate our considerations with an example!

Example 1.1. Let $X = S^1 \subset \boldsymbol{C}$, be the unit sphere, and let $A = \{e^{k\pi i/2} \mid k = 0, 1, 2, 3\}$. We define self maps f and g of (X, A) as follows. Define f on the interval $[0, \pi]$ by $f(e^{i\theta}) = e^{2i\theta}$, and on $[\pi, 2\pi]$ by $f(e^{i\theta}) = e^{-2i\theta}$. Let $g = f^2$, the second iterate of f. There are four coincidence points $\{e^{k\pi i/3} \mid k = 0, 2, 3, 4\}$, two in A, and two in the complement $X - A$. Now $N(f, g; X, A) = 2$, however, as we shall see, any maps $f_1 \simeq f$ and $g_1 \simeq g$ (relative homotopies) will have at least two coincidences on A and two on $X - A$.

The aim for both parts of this paper is to determine the minimum number $M(f, g; X - A)$ of coincidence points (under relative homotopies of f and g) on the complement $X - A$. Since in the example $M(f, g; X - A) = 4$, and $N(f, g; X, A) = 2$ then this number is seen at times to be inadequate. Also the number $N(f, g; X, A)$ does not in general give us the location of the coincidences. It turns out the appropriate way to study our question depends on whether or not the subspace A can be by-passed. Thus the division of the paper into two parts is natural.

In the first part we have in view that the subspace can indeed be by-passed. We deal here then with ordinary Nielsen coincidence classes, and define a Nielsen type number $N(f, g; X - A)$, that will detect which of the $N(f, g)$ ordinary Nielsen classes lie entirely in the complement, and none of whose representatives can be moved to A by homotopies of f and g. As usual in Algebraic Topology the geometry is reflected in algebra. In this case it is reflected in the algebra of Reidemeister classes. In the presence of Jiang type conditions this enables us to make calculations. We prove a Minimum Theorem 3.23, which requires that A satisfies the by-passing condition. This first part of the paper (and the new Jiang type condition in Section 2) is based on part of the first author's thesis [5].

When A cannot be by-passed the minimum theorem mentioned above does not hold. We deal with this scenario, and in particular with Example 1.1, in the second part of the paper. Note in Example 1.1 that all four points are in the same Nielsen class of f and g. This being the case it should be clear that it is not enough to simply deal with ordinary Nielsen classes. We define and study a new Nielsen type number $SN(f, g; X - A)$ which uses surplus (and nonsurplus) classes. These are a kind of local (and therefore finer) coincidence point class defined on the subspace $X - A$. We show that $SN(f, g; X - A) \geqslant N(f, g; X - A)$, and that equality holds (and thus the two theories coincide) when A can be by-passed. We also show that this number satisfies an appropriate Minimum Theorem 4.10.

We have suggested that the questions raised in this paper are natural, and of interest in their own right. However, for us an important additional motivation for the first part of this paper is that it serves as preparation for work on equivariant coincidence theory ([7,8], see also Example 3.18 which gives insight into how relative theory might be connected to equivariant theory). In particular, we will want to use Jiang type theorems for calculations there (cf. Theorem 3.12 and Corollary 3.14). Since this is a primary focus we have not attempted to give the results in full generality. For example, we consider neither (a) the semi-index of Jezierski (see [13]), nor do we (b) extend our consideration to manifolds with boundary following the work of [4]. In fact, as shown in [5], in the latter case most things described in this paper go through.

After the first author's thesis was submitted [5], and the work on this paper substantially completed, the paper [14] came to our attention. That paper sketches some of the results of section three (for example, Theorems 3.9, 3.10 and 3.12). However, as preparation for [8] the details in [14] are inadequate. For example, the result [14, Theorem 3.5], that corresponds to our main computational Theorem 3.12, is wrong when the subspace is not connected. This is because the shift defined in our Lemma 3.11 is missing—see Example 3.13 for more details. In addition the applications of our new Jiang type condition are absent from [14], there is no minimum theorem to indicate the status of $N(f, g; X - A)$ as a lower bound, and there are no examples. Of course, [14] does not deal with the second part of this work either.

Other features of this work include a modified fundamental group approach to Reidemeister classes, and our exposition differs for that of [18] and [19] in that our section three is designed to minimize the details needed for Section 4 (compare the proofs of Theorems 3.23 and 4.10). The modified fundamental group approach has been used in fixed point theory (see [9,10] etc), but not previously in coincidence theory. Contrary to the comments in [11, p. 2] this approach has all the advantages of the covering space approach in that we can discuss possibly empty classes (very necessary for this paper), but without bringing in the machinery of covering spaces. This considerably simplifies the exposition.

The paper is divided as follows. Following this introduction we have in section two a kind of review of ordinary coincidence theory and relative coincidence theory. We say a kind of review, because parts of it are new, and we use the modified fundamental group approach. In section three the theory of ordinary coincidence classes on the complement are dealt with, while in section four we deal with surplus coincidence theory.

2. (Local) coincidence theory

In this section we rework and extend the concepts needed in the study of (local) Nielsen coincidence theory for index. For the algebraic side we use an amended fundamental group approach which parallels the approach for fixed point theory introduced in [9]. This approach which assigns an index to the Reidemeister classes (as well as the geometric classes) enable us to discuss possibly empty classes thus eliminating the supposed advantage of the covering space approach. Throughout the paper unless otherwise stated the spaces X and Y will be compact closed oriented manifolds of the same dimension.

2.1. Nielsen coincidence classes

There are two sides to the theory: the algebraic and the geometric. The geometric portion of the theory concerns itself with the Nielsen equivalence of coincidence points, and of index of the resulting Nielsen classes. Let X and Y be manifolds of the same dimension, and $V \subseteq X$ an open subspace of X which has finitely many path components. Suppose that $f, g: V \to Y$ are maps, we let $\Phi_V(f, g) = \{x \in V\colon\ f(x) = g(x)\}$ denote the set of geometric coincidences of f and g on V. We say that $x, y \in \Phi_V(f, g)$ are Nielsen equivalent provided that there is a path c from x to y in V such that $f(c) \simeq g(c)$ rel end points. We write $f(c) \sim g(c)$ for short. The set of equivalence classes thus generated are denoted by $\widetilde{\Phi}_V(f, g)$. We call them the set of all geometric Nielsen coincidence classes of f and g (on V). When $V = X$ we will omit the V and write $\Phi_X(f, g)$ as $\Phi(f, g)$ etc. Throughout the paper we will use designations such as $[x]_X$, $[x]_V$ or simply $[x]$ to denote generic geometric Nielsen coincidence classes of f and g.

The next step is to assign an index to (some of) the geometric Nielsen classes $[x] \in \widetilde{\Phi}_V(f, g)$. The definition of the index of a Nielsen coincidence point class $[x]$ is standard when it is compact (see, for example, [15,1] or [17]). This compactness is automatic for the two applications we have in mind namely (a) all (ordinary) Nielsen classes for the case X is compact and $V = X$, and (b) the surplus coincidence point classes which occur when X and A are compact and $V = X - A$ (see Theorem 4.3). A compact class $[x]$ on V or X is said to be essential if its index is nonzero. The number $N(f, g)$ denotes the number of essential coincidences classes of f and g on X.

It is important in Nielsen theory that the numbers defined are homotopy invariants. We introduce some notation here to accommodate this. Let $H: h_0 \simeq h_1: V \times I \to Y$ be a homotopy, and $c: a \to b$ a path in V. We define paths $H(c)$ and $H^{-1}(c)$ in Y by $H(c)(t) = H(c(t), t)$, and $H^{-1}(c)(t) = H(c)(1 - t)$, respectively. If $x_0 \in V$, then $H(x_0)$ will denote the path $H(d)$, where d is the constant path at x_0. If $F: f \simeq f_1$ and $G: g \simeq g_1$ are homotopies then we say that $x \in \Phi_V(f, g)$ is F,G related to $x' \in \Phi_V(f_1, g_1)$ if there is a path $\delta: x \to x'$ in V such that $F(\delta) \sim G(\delta)$. This relation extends to coincidence point classes.

Theorem 2.1 (See, for example, [1, Theorem 24, p. 81]). *If $[x] \in \widetilde{\Phi}_V(f, g)$ is F,G related to $[x'] \in \widetilde{\Phi}_V(f_1, g_1)$, then the index of $[x]$ is equal to the index of $[x']$. Moreover if for $[x]$ as above there is no $[x'] \in \widetilde{\Phi}_V(f_1, g_1)$ to which $[x]$ is F,G related, then the index of $[x]$ is zero.*

2.2. Reidemeister classes, modified fundamental group approach

From now on in this section we assume that $V = X$, and that maps $f, g: X \to Y$ are given. In what follows we shall not distinguish between a path and its path class in the fundamental groupoid $\pi(X)$ (or $\pi(Y)$). Thus c can denote both a path and a path class in $\pi(X)$. In addition if $h : X \to Y$ is a map, $h(c)$ will denote either a path or class. If $c : a \to b$ is a path, then $c^{-1} : b \to a$ is the path defined by $c^{-1}(t) = c(1-t)$.

We choose base points $x_0 \in X$, $y_0 \in Y$, but we do not assume that either f or g is base point preserving. So we choose paths ω from y_0 to $f(x_0)$ and μ from y_0 to $g(x_0)$. Using the paths ω and μ we define homomorphisms f_*^ω and g_*^μ on $\pi_1(X, x_0)$ by $f_*^\omega(\alpha) = \omega f(\alpha)\omega^{-1}$, and $g_*^\mu(\alpha) = \mu g(\alpha)\mu^{-1}$. If $y_0 = f(x_0)$, and ω is the constant path, then we shall omit the ω and write f_*. The homomorphisms f_*^ω and g_*^μ determine an equivalence relation on $\pi_1(Y, y_0)$ ($f_*^\omega - g_*^\mu$ congruence) defined by the rule that $\alpha \sim \beta$ in $\pi_1(Y, y_0)$ if and only if there exists $\gamma \in \pi_1(X, x_0)$ with $\alpha = g_*^\mu(\gamma)\beta f_*^\omega(\gamma^{-1})$. The resulting classes are called Reidemeister classes. The Reidemeister class containing α is denoted by $[\alpha]$. The set of all Reidemeister classes is denoted by $\mathcal{R}(f_*^\omega, g_*^\mu)$. The top part of the diagram is an exact sequence of based sets,

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "g_*^\mu \cdot f_*^{-\omega}"] \arrow[d, "\theta_X"] & \pi_1(Y, y_o) \arrow[r, "j"] \arrow[d, "\theta_Y"] & \mathcal{R}(f_*^\omega, g_*^\mu) \arrow[d, "\tilde{\theta}_Y"] \\
H_1(X) \arrow[r, "g_* - f_*"] & H_1(Y) \arrow[r, "\eta_Y"] & Coker(g_* - f_*)
\end{tikzcd} \tag{1}$$

where the first function takes an element α to $g_*^\mu(\alpha) f_*^\omega(\alpha^{-1})$, and the second places an element β in it Reidemeister class $[\beta]$. Of course, θ_X and θ_Y are Hureciwz homomorphisms, and $\tilde{\theta}_Y$ the induced function. If $\pi_1(Y, y_0)$ is abelian there is a canonical group structure on $\mathcal{R}(f_*^\omega, g_*^\mu)$, moreover in this case the entire diagram consists of groups and homomorphisms, and the vertical functions are isomorphisms.

All the above constructions are independent of the choice of base point and path classes in the sense that there exists bijections between the various Reidemeister sets (see [9] for the fixed point case). For example:

Proposition 2.2. *Let f, g, x_0, y_0, ω and μ be as above, and let μ_1 and ω_1 be other paths from y_0 to $g(x_0)$ and $f(x_0)$, respectively. Then there is a bijection*

$$\Psi : \mathcal{R}(f_*^\omega, g_*^\mu) \to \mathcal{R}(f_*^{\omega_1}, g_*^{\mu_1})$$

defined by $\Psi([\alpha]) = [\mu_1\mu^{-1}\alpha\omega\omega_1^{-1}]$.

The cardinality $\#(\mathcal{R}(f_*^\omega, g_*^\mu))$ of $\mathcal{R}(f_*^\omega, g_*^\mu)$ is called the Reidemeister number and is denoted by $R(f, g)$. From the above discussion, $R(f, g)$ is independent of the choice of base points x_0 and y_0, and of the paths ω and μ.

As an example if $X = Y = S^1$, f and g are the standard maps of degree n and m, and ω and μ are the constant paths at 0, then from the exact sequence above $\mathcal{R}(f_*^\omega, g_*^\mu) \cong \mathbb{Z}_{|m-n|}$, so $R(f, g) = |m - n|$.

2.3. Relations between Reidemeister and Nielsen classes

In this subsection we define the relationship between geometric and Reidemeister classes, we then define index for Reidemeister classes. This is the heart of the modifications of the fundamental group approach mentioned earlier. We then discuss computation and introduce a new Jiang type condition to help in the calculation of $N(f,g)$.

The following injective function establishes a relationship between the geometric and Reidemeister classes. Define

$$\rho = \rho_{\omega,\mu} : \widetilde{\Phi}(f,g) \to \mathcal{R}(f_*^{\omega}, g_*^{\mu}),$$

on a class $[x]$ by choosing a path $c : x_0 \to x$ and defining $\rho([x]) = [\mu g(c) f(c^{-1})\omega^{-1}]$. The function ρ is well defined and is an injection. Thus

$$N(f,g) \leqslant R(f,g).$$

Generalizing the fixed point case in [9] we next define index of a Reidemeister class. Let $[\alpha] \in \mathcal{R}(f_*^{\omega}, g_*^{\mu})$, we define

$$i[\alpha] = \begin{cases} 0 & \text{if } [\alpha] \notin \mathrm{Im}(\rho), \\ \mathrm{ind}([x]) & \text{if } \rho([x]) = [\alpha]), \end{cases}$$

where ind is the index given in Section 2.1. All this is independent of the choices of x_0, y_0, μ and ω in the sense that for different choices, the canonically defined bijections between the Reidemeister sets commute with the ρ functions. For example, $\Psi\rho = \rho$ in Proposition 2.2 (see [9] for details of the fixed point case). It follows that these bijections are index preserving.

If $F : f \simeq f_1$ and $G : g \simeq g_1$ are homotopies, then there is a bijection

$$\Theta = \Theta_{F,G} : \mathcal{R}(f_*^{\omega}, g_*^{\mu}) \to \mathcal{R}\big(f_{1*}^{\omega F(x_0)}, g_{1*}^{\mu G(x_0)}\big).$$

defined by $\Theta([\alpha]) = [\alpha]$.

Proposition 2.3. *The function Θ is index preserving.*

Proof. We show that if $[x] \in \widetilde{\Phi}(f,g)$ is F,G related to $[x]' \in \widetilde{\Phi}(f_1,g_1)$, then $\Theta(\rho([x])) = \rho([x'])$. The result then follows from Theorem 2.1. Let $\delta : x \to x'$ be a path in X such that $F(\delta) \sim G(\delta)$, and let $c : x_0 \to x$. Then $c\delta$ is a path from x_0 to x'. Note that

$$\begin{aligned} &\mu G(x_0) g_1(c\delta) f_1(\delta^{-1}c^{-1}) F^{-1}(x_0)\omega^{-1} \\ &\quad = \mu G(x_0) g_1(c) g_1(\delta) f_1(\delta^{-1}) f_1(c^{-1}) F(x_0)\omega^{-1} \\ &\quad = \mu g(c) G(\delta) F^{-1}(\delta) f(c^{-1})\omega^{-1} \\ &\quad = \mu g(c) f(c^{-1})\omega^{-1}. \end{aligned}$$

So $\rho([x']) = \Theta(\rho([x]))$ as required. $\square$

Thus in this modified fundamental group approach we essentially identify a geometric class (empty or not) with its "coordinate" (including index).

Theorem 2.4. *If $F : f \simeq f_1$ and $G : g \simeq g_1$ are homotopies, then $N(f,g) = N(f_1,g_1)$.*

Let $M(f,g)$ be the number

$$M(f,g) = \min \#\big(\{\Phi(f_1,g_1) \mid f \simeq f_1,\ g \simeq g_1\}\big).$$

Theorem 2.5. $N(f,g) \leqslant M(f,g)$.

We can say more when the homotopies F and G above are self homotopies of f, respectively g. Define $\langle F, G\rangle \in \pi_1(Y, y_0)$ to be $\omega F(x_0)\omega^{-1}\mu G^{-1}(x_0)\mu^{-1}$. Let

$$T(f_*^\omega, g_*^\mu) = \bigl\{\langle F, G\rangle \mid F : f \simeq f,\ G : g \simeq g\bigr\} \subseteq \pi_1(Y, y_0),$$

and let $\widetilde{T}(f_*^\omega, g_*^\mu)$ be the image of $T(f_*^\omega, g_*^\mu)$ under projection from $\pi_1(Y, y_0)$ to $\mathcal{R}(f_*^\omega, g_*^\mu)$. Then any two elements of $\widetilde{T}(f_*^\omega, g_*^\mu)$ are F, G related for some F and G, and so have the same index. Let $J(f_*^\omega)$ denote the subset of $T(f_*^\omega, g_*^\mu)$ that has each $G : g \simeq g$ the constant homotopy, and let $\tilde{J}(f_*^\omega)$ denote the corresponding image in $\mathcal{R}(f_*^\omega, g_*^\mu)$. Then

$$\tilde{J}(f_*^\omega) \subseteq \widetilde{T}(f_*^\omega, g_*^\mu) \subseteq \mathcal{R}(f_*^\omega, g_*^\mu).$$

Definition 2.6. Let $f, g : X \to Y$ be maps, we say that f and g are *pseudo Jiang* if g_*^μ is onto and $f_*^\omega(\pi_1(Y, y_0)) \subseteq J(f_*^\omega)$.

Recall that Y is a Jiang space if any $\alpha \in \pi_1(Y, y_0)$ can be represented by a loop $H(y_0)$, for some self homotopy H of the identity on Y. Note that if Y is a Jiang space this does not imply that any two maps f and g into Y are pseudo Jiang, since g_*^μ might not be onto.

The essence of the proof of the first part of the next theorem, which is well known (see [1]), is to observe when Y is a Jiang space that $J(f_*^\omega) = \pi_1(Y)$, so (1) $\tilde{J}(f_*^\omega) = \widetilde{T}(f_*^\omega, g_*^\mu) = \mathcal{R}(f_*^\omega, g_*^\mu)$, and that in this case $\pi_1(Y)$ is abelian, so (2) $\tilde{\theta}_Y$ of diagram (1) is a bijection. We use this in the proof of the second part of the next theorem which (together with its illustration) is new.

From now on we will assume that Y is an orientable manifold (so that the Lefschetz number $L(f,g)$ is defined).

Theorem 2.7. *If Y is a Jiang space, or if f and g are pseudo Jiang, then $\tilde{\theta}_Y : \mathcal{R}(f_*^\omega, g_*^\mu) \to \mathit{Coker}(g_* - f_*)$ is an isomorphism of groups and*

$$N(f,g) = \begin{cases} 0 & \text{if } L(f,g) = 0, \\ \#\mathit{Coker}(g_* - f_*) & \text{if } L(f,g) \neq 0\,. \end{cases}$$

For the definition of $L(f,g)$ see, for example, [17].

Proof. The proof follows in parallel the two steps outlined just prior to the statement of the theorem. For (1) it is sufficient to prove that for any $\alpha \in \pi_1(Y, y_0)$, there is $\gamma \in \pi_1(X, x_0)$ such that $\alpha \sim f_*^\omega(\gamma)$. Since g_*^μ is onto, there is $\gamma \in \pi_1(X, x_0)$ such that $g_*^\mu(\gamma) = \alpha$. Now $\alpha \sim g_*^\mu(\gamma^{-1})\alpha f_*^\omega(\gamma) = f_*^\omega(\gamma)$.

For (2) we give the proof in several steps.

Step I. For any $\alpha, \beta, \gamma \in \pi_1(Y, y_0)$, $\alpha\beta\gamma \sim \beta\alpha\gamma$. Note that $f_*^\omega(\pi_1(X, x_0)) \subseteq J(f^\omega)$ implies that $f_*^\omega(\pi_1(X, x_0))$ is abelian. Let $\alpha, \beta, \gamma \in \pi_1(Y, y_0)$, since g_*^μ is onto, we can choose $a, b, r \in \pi_1(X, x_0)$ such that $g_*^\mu(a) = \alpha$, $g_*^\mu(b) = \beta$ and $g_*^\mu(r) = \gamma$. Then we have

$$\begin{aligned}\alpha\beta\gamma &\sim g_*^{\mu}\big((abr)^{-1}\big)(\alpha\beta\gamma) f_*^{\omega}(abr) = f_*^{\omega}(abr) = f_*^{\omega}(a) f_*^{\omega}(b) f_*^{\omega}(r)\\ &= f_*^{\omega}(b) f_*^{\omega}(a) f_*^{\omega}(r) = f_*^{\omega}(bar) \sim g_*^{\mu}(bar) f_*^{\omega}(bar) f_*^{\omega}\big((bar)^{-1}\big)\\ &= g_*^{\mu}(bar) = g_*^{\mu}(b) g_*^{\mu}(a) g_*^{\mu}(r) = \beta\alpha\gamma.\end{aligned}$$

Step II. For any commutator $[\alpha,\beta]$ and any γ in $\pi_1(Y,y_0)$, $[\alpha,\beta]\gamma \sim \gamma$. By Step I, $[\alpha,\beta]\gamma = \alpha\beta(\alpha^{-1}\beta^{-1}\gamma) \sim \beta\alpha(\alpha^{-1}\beta^{-1}\gamma) = \gamma$.

Step III. If $\theta_Y(\gamma) = \theta_Y(\gamma')$, then $\gamma \sim \gamma'$. In fact, in this case, $\gamma'\gamma^{-1} \in \operatorname{Ker}\theta$, the commutator subgroup. Therefore, $\gamma' = [\alpha_1,\beta_1]\cdots[\alpha_k,\beta_k]\gamma$ for some α_i, β_i. Repeating Step II, we have $\gamma' \sim \gamma$.

Finally we have Step IV. This ensures that different Reidemeister classes are taken to distinct Reidemeister classes. Surjectivity is automatic.

Step IV. If $\eta\circ\theta_Y(\beta) = \eta\circ\theta_Y(\beta')$, then $\beta \sim \beta'$. Assume $\eta\circ\theta_Y(\beta) = \eta\circ\theta_Y(\beta')$. By the definition of η, there is a $c \in H_1(X)$ such that $\theta_Y(\beta') - \theta_Y(\beta) = g_*(c) - f_*(c)$. Let $c = \theta_X(\gamma)$, where $\gamma \in \pi_1(X,x_0)$, then $g_*(c) = g_*(\theta_X(\gamma)) = \theta_Y(g_*^{\mu}(\gamma))$ and $f_*(c) = \theta_Y(f_*^{\omega}(\gamma))$. Hence, $\theta_Y(\beta') = \theta_Y(\beta) + \theta_Y(g_*^{\mu}(\gamma)) - \theta_Y(f_*^{\omega}(\gamma)) = \theta_Y(g_*^{\mu}(\gamma)\beta f_*^{\omega}(\gamma^{-1}))$. By Step III, $\beta' \sim g_*^{\mu}(\gamma)\beta f_*^{\omega}(\gamma^{-1}) \sim \beta$. $\square$

Example 2.8. Let $T = S^1\times S^1$, and let a and b be paths in $S^1\times 0$ and $0\times S^1$, respectively, which represent the generators in the fundamental group of T. We think of a and b as the standard basis of T. Let T_n be the connected sum of n copies of T, and for $i = 1,\dots,n$ let a_i, b_i be the standard basis of the ith copy of T. Now let $X = T_4$, $Y = T_2$, and define $f: X\to Y$ to be the composite $f_3f_2f_1$ where $f_1: X\to T$ projects the first factor to T and sends all else to a singleton, where $f_2: T\to S^1$ is projection on the first factor, and $f_3: S^1\to T$ sends a_1 to a_1^{-1}. Define g to be the map which sends the first two factors of X to Y by the identity, and which sends the other two factors to a single point. Then for any choices of x_0, y_0 of μ and ω g_*^{μ} is onto, $f_*^{\omega}(\pi_1(X,x_0)) = \langle a_1\rangle$, $J(f_*^{\omega}) = \langle a_1\rangle$, and $L(f,g) = 2$. By Corollary 2.7, we have

$$N(f,g) = \mathcal{R}(f,g) = \#\mathit{Coker}(g_* - f_*) = 2.$$

Example 2.9. Let X and Y and f be as in the above example. Define $g = g_3g_2g_1$ where $g_1 = f_1$, g_2 is projection to b_1, and g_3 sends S^1 to b_1. Then we have $f_*^{\omega}(\pi_1(X,x_0)) \subset J(f_*^{\omega})$ and $L(f,g) = 1$, $N(f,g) = 1$, on the other hand, as we shall now see, $R(f,g)$ is infinite. From the surjectivity of $\tilde{\theta}_Y$ in diagram (1), we need only show that $\mathit{Coker}(g_* - f_*)$ is infinite. To see this note that $H_1(Y) \cong A\oplus B$, where $A \cong B \cong \mathbb{Z}\times\mathbb{Z}$, and that the image of both f_* and g_* is contained in A. Thus $\mathit{Coker}(g_* - f_*) \cong A'\oplus B$, where A' is a quotient of A, and so $\mathit{Coker}(g_* - f_*)$ is infinite as required.

2.4. Relative coincidence theory

In this section we relate mostly the work of [13] and [12]. For the rest of this paper X, Y will be compact oriented n-dimensional manifolds, and $A \subseteq X$, $B \subseteq Y$ will be locally flat submanifolds with the same dimension m. Let $f, g: (X,A)\to(Y,B)$ be maps of pairs. We use f_A and g_A to denote the restrictions of f and g to A. Note that if $[x]_X$, and $[a]_A$ are coincidence point classes of f and g, respectively of f_A and g_A, and if $[x]_X \cap [a]_A \neq \emptyset$, then $[a]_A \subseteq [x]_X$.

Definition 2.10. Let $f, g : (X, A) \to (Y, B)$ be maps of pairs of manifolds as above. A coincidence class $[x]_X$ of $f, g : X \to Y$ is said to be a common coincidence class of f, g and f_A, g_A if it contains an essential coincidence class $[a]_A$ of f_A and g_A.

Define $N(f, g; f_A, g_A)$ to be the number of essential common coincidence classes of f, g and f_A, g_A. Clearly we have that $0 \leqslant N(f, g; f_A, g_A) \leqslant N(f, g)$, and that $0 \leqslant N(f, g; f_A, g_A) \leqslant N(f_A, g_A)$.

Definition 2.11. Let X, Y be compact oriented n-dimensional manifolds, and let $A \subseteq X$, $B \subseteq Y$ be submanifolds with the same dimension m. The relative Nielsen number $N(f, g; X, A)$ of f and g is defined to be the sum

$$N(f, g; X, A) = N(f, g) + N(f_A, g_A) - N(f, g; f_A, g_A).$$

Clearly $N(f, g; X, A) \geqslant N(f, g)$ and $N(f, g; X, A) \geqslant N(f_A, g_A)$.

Theorem 2.12 (Homotopy invariance). *If $f_0 \simeq f_1$ and $g_0 \simeq g_1$ as maps of pairs, then $N(f_0, g_0; X, A) = N(f_1, g_1; X, A)$.*

Theorem 2.13 (Lower bound property). *Any maps f_1, g_1 homotopic to $f, g : (X, A) \to (Y, B)$, respectively as maps of pairs, have at least $N(f, g; X, A)$ coincidences.*

In terms of computation Jang and Lee in [12] give a relative analogue of the first part of Theorem 2.7 for the case that both Y and B are Jiang spaces. However, since their theorem is a corollary of our considerations we defer its statement (and generalizations) until the end of Section 3.

The following theorem from [5] is a relative version of a theorem of Brooks [2]. We refer the reader to [5] or [6] for its proof.

Theorem 2.14 (A relative version of Brooks' theorem [5]). *Let $f, g : (X, A) \to (Y, B)$ be maps of a pair of topological spaces (X, A) into a pair of manifolds (Y, B), with $A \subseteq X$, $B \subseteq Y$ locally flat. Let $f \simeq f_1$ and $g \simeq g_1$ as maps of pairs. Then there is a map f_2 homotopic to f as a map of pairs such that $\Phi(f_2, g) = \Phi(f_1, g_1)$.*

There is also a relative minimum theorem [13], a version of which we give as a corollary of our considerations at the end of Section 3.

3. Coincidence points on the complement

This section is divided into three subsections, location of coincidences, computations and the minimum theorem.

3.1. Location of coincidence points

In this section we investigate coincidence points on the complement. Our aim is to find a sharp lower bound for $M(f, g; X - A)$. The machinery that we need is more subtle than that given by the ordinary relative theory outlined in the last section. If $f, g : (X, A) \to (Y, B)$, then the relative theory discussed there deals with the interaction of the essential classes of the restrictions $f_A, g_A : A \to B$ and the ordinary essential classes of the maps $f, g : X \to Y$, obtained by forgetting that f and g are actually maps of pairs. However, when it comes to the location of these coincidences (in A or in $X - A$), then we also need to consider the interaction of the nonessential (and possibly empty) classes of f_A, and g_A with the essential classes of f and g. We borrow an example from fixed point theory to illustrate our considerations.

Example 3.1. Let $f : (D^2, S^1) \to (D^2, S^1)$ be small irrational rotation. Then f has one fixed point (coincidence with the identity), and this is located in $X - A$. However, it is not hard to construct a relative homotopy of f that moves this single fixed point onto the boundary and leaves $X - A$ fixed point free.

The question for us then, in any particular example, is to determine which (if any) of the coincidence point classes of f and g can be moved to A. The criterion cannot be that it intersects with a nonempty class of f_A and g_A, since in the above example $\Phi(f_A, g_A)$ is empty! The solution to this problem introduced in [18] for the fixed point case, was to discuss "weakly common" fixed point classes. By analogy we introduce "weakly common" coincidence point classes. However, we do it in the modified fundamental group approach, discussing only the relationship between the Reidemeister "coordinates" of f and g and those of f_A and g_A, rather than (in the covering space approach as in [18]) discussing both the lifting classes and the associated Reidemeister classes of these functions.

Let $f, g : (X, A) \to (Y, B)$ be maps of pairs of spaces. Let $\hat{A} = \bigcup_1^l A_k$ be the disjoint union of all components A_k of A which are mapped by f and g into the same component (which we label) B_k of B. We shall write $f_k, g_k : A_k \to B_k$ for the restriction of f, g to A_k. If there is only one path component of A then we can choose the original ω and μ to be sitting inside A. However, in the general case we choose points $x_0 \in X$, $y_0 \in Y$, and paths ω and μ as usual. Then for each component A_k of A which is mapped by f and g into the same component B_k of B, we choose base points $a_k \in A_k$, $b_k \in B_k$ and paths $u_k : x_0 \to a_k$ in X, and $\mu_k : b_k \to g_k(a_k)$ and $\omega_k : b_k \to f_k(a_k)$ in B_k. Note the same component B_k of B may have different names with possibly different choices of b_k, ω_k and μ_k.

For each k there is a Reidemeister set $\mathcal{R}(f_{k*}^{\omega_k}, g_{k*}^{\mu_k})$ and a function

$$\tilde{v}_k = \tilde{v}_k(f, g) : \mathcal{R}(f_{k*}^{\omega_k}, g_{k*}^{\mu_k}) \to \mathcal{R}(f_*^{\omega}, g_*^{\mu})$$

which takes a class $[\beta]$ to $[v_k(\beta)]$ where $v_k : \pi_1(B_k, b_k) \to \pi_1(Y, y_0)$ is defined by $v_k(\alpha) = \mu g_k(u_k)\mu_k^{-1}\alpha\omega_k f_k(u_k^{-1})\omega^{-1}$. Note that since we identify paths and classes, we also identify the path α in B_k on the right hand side of the definition with its image $i_k(\alpha)$ in Y, where $i_k : B_k \to Y$ is the inclusion.

Lemma 3.2. *The designation* $[\alpha] \to [v_k(\alpha)]$ *is a well defined function* $\tilde{v}_k : \mathcal{R}(f_{k*}^{\omega_k}, g_{k*}^{\mu_k}) \to \mathcal{R}(f_*^{\omega}, g_*^{\mu})$. *Moreover the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{\Phi}(f_k, g_k) & \xrightarrow{\rho_k} & \mathcal{R}(f_{k*}^{\omega_k}, g_{k*}^{\mu_k}) \\ \Big\downarrow{\scriptstyle i_k} & & \Big\downarrow{\scriptstyle \tilde{v}_k} \\ \widetilde{\Phi}(f, g) & \xrightarrow{\rho} & \mathcal{R}(f_*^{\omega}, g_*^{\mu}) \end{array} .$$

Proof. Suppose that $\alpha, \beta \in \pi_1(B_k, b_k)$ are in the same Reidemeister class, i.e., there is a $\gamma \in \pi_1(A_k, a_k)$ such that $\beta = \mu_k g_k(\gamma)\mu_k^{-1}\alpha\omega_k f_k(\gamma^{-1})\omega_k^{-1}$. Then

$$\begin{aligned} v_k(\beta) &= \mu g(u_k)\mu_k^{-1}\beta\omega_k f(u_k^{-1})\omega^{-1} \\ &= \mu g(u_k)\mu_k^{-1}\mu_k g_k(\gamma)\mu_k^{-1}\alpha\omega_k f_k(\gamma^{-1})\omega_k^{-1}\omega_k f(u_k^{-1})\omega^{-1} \\ &= \mu g(u_k) g_k(\gamma)\mu_k^{-1}\alpha\omega_k f_k(\gamma^{-1}) f(u_k^{-1})\omega^{-1} \\ &= \mu g(u_k\gamma)\mu_k^{-1}\alpha\omega_k f(\gamma^{-1}u_k^{-1})\omega^{-1} \\ &= \mu g(u_k\gamma) g(u_k^{-1})\mu^{-1}\mu g(u_k)\mu_k^{-1}\alpha\omega_k f(u_k^{-1})\omega^{-1}\omega f(u_k) f(\gamma^{-1}u_k^{-1})\omega^{-1} \\ &= \mu g(u_k\gamma u_k^{-1})\mu^{-1}\big(\mu g(u_k)\mu_k^{-1}\alpha\omega_k f(u_k^{-1})\omega^{-1}\big)\omega f(u_k\gamma^{-1}u_k^{-1})\omega^{-1} \\ &= \mu g\big(i_*^{u_k}(\gamma)\big)\mu^{-1}\big(\mu g(u_k)\mu_k^{-1}\alpha\omega_k f(u_k^{-1})\omega^{-1}\big)\omega f\big(i_*^{u_k}(\gamma^{-1})\big)\omega^{-1} \\ &= g_*^{\mu}\big(i_*^{u_k}(\gamma)\big) v_k(\alpha) f_*^{\omega}\big(i_*^{u_k}(\gamma^{-1})\big). \end{aligned}$$

Thus $[v_k(\alpha)] = [v_k(\beta)]$, and $\tilde{v}_k$ is well defined.

For the second part, let $x \in [x]_k \in \widetilde{\Phi}(f_k, g_k)$, $c : a_k \to x$ be a path in A_k, and let $[x]_X = i_k([x]_k)$, where $i_k : A_k \to X$ is the inclusion. Then the diagram is commutative since

$$\begin{aligned} \tilde{v}_k\rho([x]_k) &= \big[v_k\big(\mu_k g_k(c) f_k(c^{-1})\omega_k^{-1}\big)\big] \\ &= \big[\mu g(u_k)\mu_k^{-1}\mu_k g_k(c) f_k(c^{-1})\omega_k^{-1}\omega_k f_k(u_k^{-1})\omega^{-1}\big] \\ &= \big[\mu g(u_k) g_k(c) f_k(c^{-1}) f(u_k^{-1})\omega^{-1}\big] \\ &= \big[\mu g(u_k c) f(c^{-1}u_k^{-1})\omega^{-1}\big] \\ &= \rho\big([x]_X\big). \qquad \square \end{aligned}$$

Definition 3.3. A class $[x] \in \widetilde{\Phi}(f, g)$ is said to be a weakly common coincidence class if there is a k and a $[\beta] \in \mathcal{R}(f_{k*}^{\omega_k}, g_{k*}^{\mu_k})$ such that $\rho_X([x]) = \tilde{v}_k([\beta])$. If $[x]$ is essential, it is called an essential weakly common coincidence class. The number of essential weakly common coincidence classes of f and g is denoted by $E(f, g; f_A, g_A)$.

Note that $0 \leqslant N(f, g; f_A, g_A) \leqslant E(f, g; f_A, g_A)$, and that each inequality may be strict (see [18] for examples in the fixed point case). By abuse of notation we sometimes refer to the algebraic class $\rho_X([x]) = \tilde{v}_k([\beta])$ as a weakly common coincidence class. Note from Example 3.1 that $[\beta]$ in Definition 3.3 need not be in the ρ image of $\widetilde{\Phi}(f_k, g_k)$. Thus the class $[\beta]$ may be empty. The next result generalizes Lemma 2.3 of [18].

Lemma 3.4. *A coincidence point $x \in \Phi(f,g)$ belongs to a weakly common coincidence class if and only if there is a path $\alpha:(I,0,1) \to (X,x,A)$ from x to A and a homotopy $H: g(\alpha) \simeq f(\alpha):(I,0,1) \to (Y,f(x),B)$.*

To emphasize the point we observe that $H: I \times I \to Y$ in Lemma 3.4 has $H(0,t) = f(x) = g(x)$ for all $t \in I$, but $H(1,t)$ is only required to lie in B.

Proof. Assume that x lies in a weakly common coincidence class, and let $c: I \to X$ be a path from x_0 to x. By the definitions of ρ and of weakly common coincidence class there is a component A_k of A and and an element $\beta \in \pi_1(B_k,b_k)$, such that $[\mu g(c) f(c^{-1})\omega^{-1}] = \tilde{v}_k([\beta])$. That is there is an element $\gamma \in \pi_1(X,x_0)$ such that $g_*^{\mu}(\gamma)\mu g(c) f(c^{-1})\omega^{-1} f_*^{\omega}(\gamma^{-1}) = v_k(\beta)$, or

$$\mu g(\gamma) g(c) f(c^{-1}) f(\gamma^{-1})\omega^{-1} = \mu g(u_k)\mu_k^{-1}\beta\omega_k f(u_k^{-1})\omega^{-1}.$$

From this we have $g(u_k^{-1}\gamma c) f(c^{-1}\gamma^{-1}u_k) = \mu_k^{-1}\beta\omega_k$. Note that the right hand side is contained in B_k. If we let $\alpha = c^{-1}\gamma^{-1}u_k$, it follows from the above that $g(\alpha) \simeq f(\alpha):(I,0,1) \to (Y,f(x),B)$ as required.

For the converse assume that x is a coincidence point, that c is a path from x_0 to x, and that there is a path $\alpha:(I,0,1) \to (X,x,A)$ from x to A together with a homotopy $H: g(\alpha) \simeq f(\alpha):(I,0,1) \to (Y,f(x),B)$. Let A_k be the component of A that contains $a = \alpha(1)$. The existence of H assures us that there is some component B_k of B such that both $f(A_k), g(A_k) \subseteq B_k$. We need an element $\beta \in \pi_1(B_k,b_k)$ such that $v_k(\beta) = g_*^{\mu}(\gamma)\mu g(c) f(c^{-1})\omega^{-1} f_*^{\omega}(\gamma^{-1})$ for some $\gamma \in \pi_1(X,x_0)$.

Let $c_a: a_k \to a$ be a path in A_k, and let $l = H(1,\cdot)$ be the path from $g(a)$ to $f(a)$ in B_k. Set $\gamma = u_k c_a \alpha^{-1} c^{-1}$ and $\beta = \mu_k g_k(c_a) l f_k(c_a^{-1})\omega_k^{-1}$. By assumption, we have $g(\alpha) l f(\alpha^{-1}) \sim 0$, or $l \sim g(\alpha^{-1}) f(\alpha)$. Then

$$\begin{aligned}
v_k(\beta) &= \mu g(u_k)\mu_k^{-1}\beta\omega_k f(u_k^{-1})\omega^{-1}\\
&= \mu g(u_k)\mu_k^{-1}\big(\mu_k g_k(c_a) l f_k(c_a^{-1})\omega_k^{-1}\big)\omega_k f(u_k^{-1})\omega^{-1}\\
&= \mu g(u_k) g_k(c_a) l f_k(c_a^{-1}) f(u_k^{-1})\omega^{-1}\\
&= \mu g(u_k c_a) g(\alpha^{-1}) f(\alpha) f(c_a^{-1}u_k^{-1})\omega^{-1}\\
&= \mu g(u_k c_a \alpha^{-1}) f(\alpha c_a^{-1} u_k^{-1})\omega^{-1}\\
&= \mu g(u_k c_a \alpha^{-1}) g(c^{-1})\mu^{-1}\mu g(c) f(c^{-1})\omega^{-1}\omega f(c) f(\alpha c_a^{-1} u_k^{-1})\omega^{-1}\\
&= \mu g(u_k c_a \alpha^{-1} c^{-1})\mu^{-1}\mu g(c) f(c^{-1})\omega^{-1}\omega f(c\alpha c_a^{-1} u_k^{-1})\omega^{-1}\\
&= g_*^{\mu}(\gamma)\big(\mu g(c) f(c^{-1})\omega^{-1}\big) f_*^{\omega}(\gamma^{-1}).
\end{aligned}$$

So $\tilde{v}([\beta]) = \rho([x]_X)$ as required. $\square$

There are two corollaries that are useful to us (Corollary 3.6 generalizes [18, 2.7]).

Corollary 3.5. *The definition of weakly common coincidence class is independent of the choices of x_0, y_0, a_k, b_k, u_k, ω, μ, ω_k, and μ_k.*

Corollary 3.6. *A coincidence class of (f, g) containing a coincidence point on A is a weakly common coincidence class.*

Definition 3.7. The number of essential coincidence classes of $f, g : X \to Y$, which are not weakly common coincidence classes is called the Nielsen number of f, g on the complement $X - A$. It is denoted by $N(f, g; X - A)$.

In other words, $N(f, g; X - A) = N(f, g) - E(f, g; f_A, g_A)$.

Example 3.8. In Example 1.1 f and g are homotopic, so all four points are in a single coincidence class so $N(f, g; X - A) = 0$ (but see Example 4.9).

Theorem 3.9. *For any pair of maps $f, g : (X, A) \to (Y, B)$ there are at least $N(f, g; X - A)$ coincidence points on $X - A$.*

Proof. Clearly each essential coincidence class of f and g has at least one coincidence point in X. Choose one such point in each class. If any of these points are in A, then by Corollary 3.6, the class is a weakly common coincidence class. Therefore, for each essential nonweakly common coincidence class there is at least one coincidence point in $X - A$. There are $N(f, g : X - A)$ such classes by definition. □

Theorem 3.10. *$N(f, g; X - A)$ is a homotopy invariant.*

Proof. Let $F : f \sim f_1 : (X, A) \to (Y, B)$ and $G : g \sim g_1 : (X, A) \to (Y, B)$ be homotopies. We know that $N(f, g) = N(f_1, g_1)$, so we only need to prove that if $x \in \Phi(f, g)$ and $x' \in \Phi(f_1, g_1)$ are F, G-related, and x is in a weakly common coincidence class, then so is x'.

So suppose that $[x]$ is a weakly common coincidence class, $c : x_0 \to x$ a chosen path, and $\beta \in \pi_1(B_k)$ is such that $\tilde{v}_k([\beta]) = \rho([x])$ where $\tilde{v}_k = \tilde{v}_k(f, g) : \mathcal{R}(f_{k*}^{\omega_k}, g_{k*}^{\mu_k}) \to \mathcal{R}(f_*^{\omega}, g_*^{\mu})$. As in the proof of Lemma 3.4 there exists a γ such that

$$\mu g(\gamma) g(c) f(c^{-1}) f(\gamma^{-1}) \omega^{-1} = \mu g(u_k) \mu_k^{-1} \beta \omega_k f(u_k^{-1}) \omega^{-1}.$$

We show that $\tilde{v}_k([\beta]) = \tilde{v}_k(\Theta([\beta])) = \rho([x'])$, where this time (cf. Proposition 2.3)

$$\tilde{v}_k = \tilde{v}_k(f_1, g_1) : \mathcal{R}\left(f_{1k*}^{\omega_k F_k(a_k)}, g_{1k*}^{\mu_k G_k(a_k)}\right) \to \mathcal{R}\left(f_{1*}^{\omega F(x_0)}, g_{1*}^{\mu G(x_0)}\right)$$

(cf. Corollary 3.5). Here F_k and G_k are the restrictions of F, respectively G to $A_k \times I$.

Since x and x' are F, G-related there exists a path $\delta : x \to x'$ such that $F(\delta) \sim G(\delta)$ (see Proposition 2.3). Now $v_k(\beta) = v_k(f_1, g_1)(\beta)$ is defined to be

$$\begin{aligned}
&\mu G(x_0) g_1(u_k) G_k^{-1}(a_k) \mu_k^{-1} \beta \omega_k F_k(a_k) f_1(u_k^{-1}) F^{-1}(x_0) \omega^{-1} \\
&\quad = \mu g(u_k) G_k(a_k) G_k^{-1}(a_k) \mu_k^{-1} \beta \omega_k F_k(a_k) F_k^{-1}(a_k) f(u_k^{-1}) \omega^{-1} \\
&\quad = \mu g(u_k) \mu_k^{-1} \beta \omega_k f(u_k^{-1}) \omega^{-1} \\
&\quad = \mu g(\gamma) g(c) f(c^{-1}) f(\gamma^{-1}) \omega^{-1} \text{ (from above—cf. Lemma 3.4)}
\end{aligned}$$

$$
\begin{aligned}
&= \mu g(\gamma)\mu^{-1}\big(\mu g(c) f(c^{-1})\omega^{-1}\big)\omega f(\gamma^{-1})\omega^{-1}\\
&= \mu g(\gamma)\mu^{-1}\big(\mu G(x_0) g_1(c\delta) f_1(\delta^{-1}c^{-1}) F^{-1}(x_0)\omega^{-1}\big)\omega f(\gamma^{-1})\omega^{-1}\\
&\qquad \text{(from Theorem 2.4)}\\
&= \mu g(\gamma) G(x_0) g_1(c\delta) f_1(\delta^{-1}c^{-1}) F^{-1}(x_0) f(\gamma^{-1})\omega^{-1}\\
&= \mu G(x_0) g_1(\gamma) g_1(c\delta) f_1(\delta^{-1}c^{-1}) f_1(\gamma^{-1}) F^{-1}(x_0)\omega^{-1}\\
&= \mu G(x_0) g_1(\gamma c\delta) f_1(\delta^{-1}c^{-1}\gamma^{-1}) F^{-1}(x_0)\omega^{-1}.
\end{aligned}
$$

Since $\rho([x'])$ is independent of the choice of path from $x_0 \to x'$ (in this case $\gamma c\delta$), then the last loop represents the class of $\rho([x'])$, and so $\tilde{v}_k([\beta]) = \rho([x'])$ as required. □

3.2. The computation of the Reidemeister and Nielsen numbers over the complement

In this section, we generalize the results in Section 4 of [18]. As before we let $f, g : (X, A) \to (Y, B)$ be a pair of maps, and $\hat{A} = \bigcup_1^{\ell} A_k$ the disjoint union of the components A_k of A which are mapped by f and g into the same component B_k of B. For each element A_k of $\hat{A}$ there is a commutative diagram

$$
\begin{tikzcd}
\pi_1(B_k, b_k) \arrow[r, "\theta_k"] \arrow[d, "i_{k*}"] & H_1(B_k) \arrow[r, "\eta_k"] \arrow[d, "i_{k*}"] & \mathit{Coker}(g_{k*} - f_{k*} : H_1(A_k) \to H_1(B_k)) \arrow[d, "i_{k*}"] \\
\pi_1(Y, b_k) \arrow[r, "\theta_Y"] & H_1(Y) \arrow[r, "\eta_Y"] & \mathit{Coker}(g_* - f_* : H_1(X) \to H_1(Y))
\end{tikzcd},
$$

where by abuse of notation the i_{k*} are induced by inclusions $i_k : A_k \to X$ or $i_k : B_k \to Y$. Let $\gamma_k = \mu g(u_k)\mu_k^{-1}\omega_k f(u_k^{-1})\omega^{-1}$, and define $\sigma_k : H_1(B_k) \to H_1(Y)$ by $\sigma_k(\alpha) = i_{k*}(\alpha) + \theta_Y(\gamma_k)$. Then σ_k induces a function $\tilde{\sigma}_k : \mathit{Coker}(g_{k*} - f_{k*}) \to \mathit{Coker}(g_* - f_*)$. Note that $\tilde{\sigma}_k$ is a homomorphism if and only if $\theta_Y(\gamma_k) = 0$.

Lemma 3.11. *The function $\tilde{\sigma}_k$ is well defined, and the following induced diagram is commutative* (*see diagram* (1)).

$$
\begin{tikzcd}
\mathcal{R}(f_{k*}^{\omega_k}, g_{k*}^{\mu_k}) \arrow[r, "\tilde{\theta}_k"] \arrow[d, "\tilde{v}_k"] & \mathit{Coker}(g_{k*} - f_{k*}) \arrow[d, "\tilde{\sigma}_k"] \\
\mathcal{R}(f_*^{\omega}, g_*^{\mu}) \arrow[r, "\tilde{\theta}_Y"] & \mathit{Coker}(g_* - f_*)
\end{tikzcd}.
$$

Proof. That $\tilde{\sigma}_k$ is well defined follows from the commutativity of the diagram at the beginning of this subsection. For the second part we show that $\sigma_k\theta_k = \theta_Y v_{f_k, g_k} : \pi_1(B_k, b_k) \to H_1(Y)$. We again use the commutativity of the diagram at the beginning of this subsection. Let $\gamma_k' = \mu g(u_k)\mu_k^{-1}$ and $\gamma_k'' = \omega_k f(u_k^{-1})\omega^{-1}$. Then for each $\alpha \in \pi_1(B_k, b_k)$, we have

$$
\begin{aligned}
\theta_Y v_k(\alpha) &= \theta_Y\big(\gamma_k' i_{k*}(\alpha)\gamma_k''\big) = \theta_Y(\gamma_k') + \theta_Y\big(i_{k*}(\alpha)\big) + \theta_Y(\gamma_k'')\\
&= \theta_Y\big(i_{k*}(\alpha)\big) + \theta_Y(\gamma_k') + \theta_Y(\gamma_k'') = i_{k*}\theta_k(\alpha) + \theta_Y(\gamma_k'\gamma_k'')\\
&= i_{k*}\theta_k(\alpha) + \theta_Y(\gamma_k) = \sigma_k\big(\theta_k(\alpha)\big). \qquad \square
\end{aligned}
$$

Case (a) of the next theorem corrects [14, Theorem 3.5] (see Example 3.13).

Theorem 3.12. *Let $f, g : (X, A) \to (Y, B)$ be maps, and suppose either* (a) *that Y is a Jiang space or* (b) *that f and g are pseudo Jiang. If $L(f, g) = 0$ then $N(f, g) = 0$; if $L(f, g) \neq 0$ then*

$$N(f, g; X - A) = \#\,\mathit{Coker}(g_* - f_*) - \#\left\{\bigcup_{k=1}^{\ell} \tilde{\sigma}_k\, \mathit{Coker}(g_{k*} - f_{k*})\right\}.$$

Proof. If $L(f, g) = 0$, then by Theorem 2.7 all the coincidence classes have zero index, so $N(f, g; X - A) = 0$. If $L(f, g) \neq 0$, all the coincidence classes will have nonzero index and the total number of coincidence classes is $\mathit{Coker}(g_* - f_*)$ by Theorem 2.7. The result follows from the definition of weakly common classes, from Lemma 3.11 and the fact that $\tilde{\theta}_k$ and $\tilde{\theta}_Y$ are isomorphisms (see Theorem 2.7). □

Example 3.13. An example of Zhao [18, 4.10] is instructive. It shows Theorem 3.12 is wrong in general if $\bigcup_{k=1}^{\ell} \tilde{\sigma}_k\, \mathit{Coker}(g_{k*} - f_{k*})$ is replaced (as in [14, 3.5]) by $\bigcup_{k=1}^{\ell} i_{k*}\, \mathit{Coker}(g_{k*} - f_{k*})$. The example is for f the flip map on S^1 ($f(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{-\mathrm{i}\theta}$) and g the identity. Here $A = \{\pm 1\}$ is the fixed point set of f. Clearly there are no coincidence points on the complement, and

$$\mathit{Coker}(g_* - f_*) = \mathbb{Z}_2 = \bigcup_{k=1}^{2} \tilde{\sigma}_k\, \mathit{Coker}(g_{k*} - f_{k*})$$

as Zhao shows (see [18, 4.10]). However, the i_{k*} are homomorphisms, and $H_1(A_k) = \{0\}$, so $\bigcup_{k=1}^{2} i_{k*}\, \mathit{Coker}(g_{k*} - f_{k*}) = \{0\}$. Note that $L(f, g) = L(f) \neq 0$, thus $N(f, g; X - A) = 0$, but from above $\#(\mathit{Coker}(g_* - f_*) - \bigcup_{k=1}^{2} i_{k*}\, \mathit{Coker}(g_{k*} - f_{k*})) = 1$.

Corollary 3.14. *Suppose A is connected and either Y is a Jiang space, or that f and g are pseudo Jiang. If $L(f, g) = 0$, then $N(f, g; X - A) = 0$; and if $L(f, g) \neq 0$, then*

$$N(f, g; X - A) = \#\,\mathit{Coker}(g_* - f_*) - \#\big(\tilde{\sigma}_A\, \mathit{Coker}(g_{A*} - f_{A*})\big).$$

Example 3.15. Let $X = Y = S^1 \times S^1 \times S^2$, and let $A = B = \Delta S^1 \times S^1$, where ΔS^1 is the diagonal in $S^1 \times S^1$, and the second S^1 in A is the equator in S^2. Define $f, g : X \to Y$ as follows:

$$f\big(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\phi}, (x, y, z)\big) = \big(\mathrm{e}^{2\mathrm{i}\phi}, \mathrm{e}^{2\mathrm{i}\theta}, (x, y, z)\big), \quad \text{and}$$
$$g\big(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\phi}, (x, y, z)\big) = \big(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\phi}, (x, -y, -z)\big).$$

There are six coincidence points and three coincidence classes of f and g in X, while there are two coincidence points and two classes of f_A and g_A in A. Since X and A are Jiang spaces, and $L(f, g) \neq 0$ and $L(f_A, g_A) \neq 0$, then $N(f, g) = \#(\mathit{Coker}(g_* - f_*)) = 3$, and $N(f_A, g_A) = \#(\mathit{Coker}(g_{A*} - f_{A*})) = 2$. We take $x_0 = a_A$ to be one of the coincidence points of A, $y_0 = b_A = f(x_0)$, we may then take μ, ω, μ_A, ω_A and u_A to be constant paths. In this case γ defined at the beginning of this section is constant, and so $\tilde{\sigma}_A = i_{A*}$ is a homomorphism of $\mathbb{Z}_2$ to $\mathbb{Z}_3$. Thus $\#(\tilde{\sigma}_A\, Coker(g_{A*} - f_{A*})) = 1$, and $N(f, g; X - A) = 3 - 1 = 2$. In particular, the two coincidence point classes in $X - A$ cannot be moved to A by relative homotopies of f and g.

Theorem 3.16. *Let $f, g : (X, A) \to (Y, B)$ be a pair of maps, and $\hat{A} = \bigcup_1^{\ell} A_k$ the disjoint union of the components A_k of A which are mapped by f and g into the same component B_k of B. Suppose further that either Y and B are Jiang spaces, or f, g and f_k, g_k are pseudo Jiang for $1 \leqslant k \leqslant \ell$. If $L(f, g) \cdot \prod_{k=1}^{m} L(f_k, g_k) \neq 0$ and $L(f_k, g_k) = 0$ for $m < k \leqslant \ell$, then the relative Nielsen number $N(f, g; X, A)$ is given by*

$$N(f, g; X, A) = \#\, Coker(g_* - f_*) + \sum_{k=1}^{m} \#\, Coker(g_{k*} - f_{k*}) - \#\left\{ \bigcup_{k=1}^{m} \tilde{\sigma}_k\, Coker(g_{k*} - f_{k*}) \right\}.$$

Proof. By definition of $N(f, g; X, A)$, and by Theorem 2.7,

$$N(f, g; X, A) = \#\, Coker(g_* - f_*) + \sum_{k=1}^{m} \#\, Coker(g_{k*} - f_{k*}) - N(f, g; f_A, g_A).$$

Now an essential coincidence class of f and g is a common coincidence class if and only if it contains an essential coincidence class of f_k and g_k for some k. By Theorem 2.7 and the hypothesis this is the case if and only if $1 \leqslant k \leqslant m$. Thus $N(f, g; f_A, g_A) = \#\{\bigcup_{k=1}^{m} \tilde{\sigma}_k\, Coker(g_{k*} - f_{k*})\}$ as required. □

As in Example 3.15 we may without loss of generality take $\tilde{\sigma} = i_{A*}$ in the next corollary. Part (i) can also be found in [12].

Corollary 3.17. *Let A be connected and $L(f, g) \cdot L(f_A, g_A) \neq 0$. Suppose further that at least one of the following hold*:

(i) *Y and B are Jiang spaces,*
(ii) *Y is an Jiang space and f_A and g_A are pseudo Jiang,*
(iii) *f and g are pseudo Jiang, and B is a Jiang space,*
(iv) *both f and g, and f_A and g_A are pseudo Jiang. Then*

$$N(f, g; X, A) = \#\, Coker(g_* - f_*) + \#\big(Coker(g_{A*} - f_{A*})\big) - \#\big(i_{A*}\, Coker(g_{A*} - f_{A*})\big).$$

Proof. Note that under the conditions of the corollary we can chose $x_0 = a_A$ to be a coincidence point, $y_0 = b_A = f(x_0) = g(x_0)$, and all the paths u_A, ω, μ, ω_A, and μ_A constant. Thus as in Example 3.15, $\tilde{\sigma}_A = i_{A*}$ is a homomorphism. The result follows. □

We conclude this subsection with an example which tentatively shows the connection between equivariant and relative coincidence theory.

Example 3.18. Let $X = Y = S^2$ the unit two sphere in $\mathbb{R}^3$. We denote points in $\mathbb{R}^3$ by cylindrical coordinates (r, θ, z). Let $W = \mathbb{Z}_2 = \langle \alpha \rangle$, and consider the action of W on S^2 determined by $\alpha \cdot (1, \theta, z) = (1, \theta, -z)$. A map f is said to be a W map if $w \cdot f(x) = f(w \cdot x)$ for all $x \in X$ and $w \in W$. Note that the fixed point set X^W of the action is given by $X^W = S^1$ the equator in S^2. Let $f, g : X \to X$ be the W maps defined by $f(1, \theta, z) = (1, -\theta, -z)$, and $g(1, \theta, z) = (1, 3\theta, z)$, respectively. Since S^2 is simply connected and the Lefschetz number $L(f, g) \neq 0$, we have from ordinary Nielsen coincidence theory that $N(f, g) = 1$. Note that $\Phi(f, g) = \{(1, \theta, 0) \mid \theta \in \{0, \pi/2, \pi, 3\pi/2\}\}$, i.e., $\#(\Phi(f, g)) = 4$, and that f and g restrict to maps $f^W, g^W : X^W \to Y^W$. In fact any f_1 and g_1 which are W homotopic to f and g, respectively, restrict to maps $f_1^W, g_1^W : X^W \to Y^W$ which are homotopic to f^W and g^W, respectively. Thus there are at least $N(f, g; X, X^W) = 4$ coincidence points for any such f_1 and g_1.

Example 3.18 is not typical of the way that relative coincidence theory is used in equivariant coincidence theory. A more typical use is the theory on the complement (see Example 3.15 the motivation for which comes in fact from equivariant theory). Example 3.18 does however show the existence of a connection however tentative.

3.3. Minimum theorems

The proof of our first lemma is contained in the proof of [13, 2.4]. Note that $U \subset (X - A)$ in that reference. In order to use this as a relative result we need to choose U this way too.

Lemma 3.19. *Let X, Y be manifolds with the same dimension $\geqslant 3$, and let $f, g : X \to Y$ be maps with a finite number of coincidence points. Let $x_0, x_1 \in \Phi(f, g)$ and α a path from x_0 to x_1 such that $f(\alpha) \sim g(\alpha)$ and $\alpha((0, 1)) \cap \Phi(f, g) = \emptyset$. Let U be a neighborhood of $\alpha([0, 1))$ such that $\overline{U} \cong D^n$ and $x_1 \in \partial \overline{U}$. Then are maps $f_1, g_1 : X \to Y$ with $f_1 \simeq f$ rel $X - U$ and $g_1 \sim g$ rel $X - U$, and such that $\Phi(f_1, g_1) = \Phi(f, g) - \{x_0\}$.*

We need a more precise formulation of weakly common coincidence class under the by-passing condition. Recall [15] that a subspace A of X if every path with end points in $X - A$ is homotopic rel end points to a path in $X - A$. The subspace in Example 1.1 cannot be by-passed.

Lemma 3.20. *Let $A \subset X$ be a submanifold such that A can be by-passed in X, $B \subset Y$ be a submanifold, and $f, g : (X, A) \to (Y, B)$ be maps. A coincidence point $x \in \Phi(f, g)$ belongs to a weakly common coincidence class if and only if there is a path*

$$\alpha : \big(I, 0, I - \{1\}, 1\big) \to (X, x, X - A, A)$$

from x to A such that $f(\alpha) \simeq g(\alpha) : (I, 0, 1) \to (Y, f(x), B)$.

Proof. The proof is similar to the proof of Lemma 3.5 in [18]. $\square$

Note that we cannot rescind the hypothesis that A can be by-passed in Lemma 3.20. To see this note in Example 1.1 that $e^{2\pi i/3}$ belongs to a weakly common coincidence class, but there is no path from $e^{2\pi i/3}$ to A of the form described in Lemma 3.20.

Lemma 3.21. *For $x \in \Phi(f,g)$, if there is a path $c:(I,0,1) \to (X,x,A_k)$ from x to A_k such that $g(c) \overset{G}{\simeq} f(c):(I,0,1) \to (Y,f(x),B)$, and if $g(c(1)) \neq f(c(1))$, then for any point $a \in A_k - \Phi(f,g)$, there exist maps f' and g' with $f' \simeq f$ and $g' \simeq g$ relative to $X - U(a)$, where $U(a)$ is a neighborhood of a in X, such that $\Phi(f',g') = \Phi(f,g) \cup \{a\}$ and $f'(\alpha) \sim g'(\alpha)$. In particular, x and a are in the same class.*

Proof. Let $l = G(1,\cdot)$, then l is a path from $g(c(1))$ to $f(c(1))$ in B. We may assume without loss of generality that $c(1) = a$. If $a \neq c(1)$, let $\alpha: I \to A_k$ be a path from $c(1)$ to a. Then since $g(c)g(\alpha)g(\alpha^{-1})lf(\alpha^{-1})f(\alpha)f(c^{-1}) \simeq g(c)lf(c^{-1}) \simeq 0$ and $g(\alpha^{-1})lf(\alpha) \subset B$, we can replace c by $c\alpha$. Let $\alpha_1,\ \alpha_2$ be paths in B such that $\alpha_1(0) = g(c(1)),\ \alpha_2(0) = f(c(1)),\ \alpha_1(1) = \alpha_2(1)$ and for any $t \neq 1$, $\alpha_1(t) \neq \alpha_2(t)$, and $\alpha_1\alpha_2^{-1} \simeq l$ rel $\{0,1\}$. Let $U(a)$ be a neighborhood of a in X, with $U(a) \cap \Phi(f,g) = \emptyset$, and such that there is a homoemorphism

$$\phi: \big(\overline{U(a)}, \overline{U(a)} \cap A_k\big) \to (D^n, D^m),$$

where D^n and D^m are the closed unit balls in $\mathbb{R}^n$ and $\mathbb{R}^m \subset \mathbb{R}^n$ with $\dim(X) = n$, and $\dim(A_k) = m$. Then using ϕ we can label each point $z \in U(a)$ by $z = (t,x)$, where $t \in I$ and $x \in \partial\overline{U(a)}$. Note that if $z = (t,x) \in A_k$ for some $t \in (0,1)$, then $z \in A_k$ for every $t \in (0,1)$. We now define f' by the formula

$$f'(z) = \begin{cases} f(z) & \text{if } z \in X - U(a),\\ f((2t-1,x)) & \text{if } z = (t,x) \in U(a) \text{ and } t \geqslant 1/2,\\ \alpha_2(1-2t) & \text{if } z = (t,x) \in U(a) \text{ and } t \leqslant 1/2. \end{cases}$$

Now $f \simeq f'$, and in fact, we can define a relative homotopy F by

$$F(s,z) = \begin{cases} f(z) & \text{if } z \in X - U(a),\\ f\left(\left(\dfrac{t-s/2}{1-s/2}, x\right)\right) & \text{if } z = (t,x) \text{ and } t \geqslant s/2,\\ \alpha_2(s-2t) & \text{if } z = (t,x) \text{ and } t \leqslant s/2. \end{cases}$$

It follows that $f'(c) \sim f(c)\alpha_2$. To see this, consider the paths $c_0(t) = (c(t),0),\ c_1(t) = (c(t),1),\ c_x(t) = (x,t),\ c_a(t) = (a,t)$ in $X \times I$. Then $c_1 \sim c_x c_0 c_a$, so $f'(c) = F(c_1) \sim F(c_x c_0 c_a) = e_{f(x)}F(c_0)F(c_a) \sim F(c_0)F(c_a) \sim f(c)\alpha_2$, where $e_{f(x)}$ is the constant path at $f(x)$. In a similar way we can construct a $g' \simeq g$ such that $g'(c) \sim g(c)\alpha_1$.

Finally using the homotopy G we see that $g(c)\alpha_1 \sim f(c)\alpha_2$, so $f'(c) \sim g'(c)$, and in particular, x and a are in the same coincidence class of $f',\ g'$ as required. □

Lemma 3.22. *Let $A \subset X$ and $B \subset Y$ be submanifolds of dimension strictly less than the dimension of X and Y, and $f,g:(X,A) \to (Y,B)$ be maps. Let x be an isolated* *coincidence point lying in* $X - A$*, and suppose that there is a path* $\alpha : (I, 0, I - \{1\}, 1) \to (X, x, X - A, A)$ *from* x *to* A *such that* $f(\alpha) \simeq g(\alpha) : (I, 0, 1) \to (Y, f(x), B)$*. If* $\dim(X) = \dim(Y) \geqslant 3$*, then there are maps* $f' \simeq f$*, and* $g' \simeq g : (X, A) \to (Y, B)$ *such that* $\Phi(f', g') = (\Phi(f, g) - \{x\}) \cup \{\alpha(1)\}$.

Proof. By Lemma 3.21 there are maps f_1, and g_1 relatively homotopic to f and g, respectively such that $\Phi(f_1, g_1) = \Phi(f, g) \cup \{\alpha(1)\}$, and $f_1(\alpha) \simeq g_1(\alpha)$. By hypothesis $\alpha([0, 1)) \subset X - A$, so we may choose U in Lemma 3.19 such that $U \subset X - A$. Thus from that lemma we obtain maps f' and g' relatively homotopic to f_1 and g_1, respectively with $\Phi(f', g') = \Phi(f_1, g_1) - \{x\} = (\Phi(f, g) - \{x\}) \cup \{\alpha(1)\}$ as required. $\square$

Theorem 3.23. *Let* $f, g : (X, A) \to (Y, B)$ *be maps of a pairs of manifolds with* $A \subseteq X$, $B \subseteq Y$ *locally flat, and suppose that* A *can be by-passed in* X*. If* $\dim(X) \geqslant 3$*, then there is a map* $f' \simeq f : (X, A) \to (Y, B)$ *such that* (f', g) *has* $N(f, g; X - A)$ *coincidence points on* $X - A$.

Proof. As in [13] (see also [5, 1.6.2]) we may assume $\Phi(f, g)$ is finite. Since A can be by-passed in X, we may assume that any points x and y in $X - A$ that lie in the same class may be joined by a path α in $X - A$ with $g(\alpha) \sim f(\alpha)$. Thus any such x and y may be coalesced in the usual way. We may therefore assume that each Nielsen class contained in $X - A$ contains at most one coincidence point. Since these classes are contained entirely in $X - A$, those classes with zero index can be removed exactly as in the absolute case. Now let $x \in X - A$ be a coincidence point, which is in a weakly common coincidence class. By Lemma 3.20 there is a path α satisfying the hypothesis of Lemma 3.22, so by that lemma x may be moved to A.

By Theorem 2.14, we can assume that $g' = g$. $\square$

Theorem 3.24. *If* $\dim A \geqslant 3$*, and* A *can be by-passed in* X*, then there is a map* $f' \simeq f$*, such that* (f, g) *has* $N(f, g; X, A)$ *coincidence points in* X *and* $N(f, g; X - A)$ *coincidence points on* $X - A$.

Proof. By Theorem 2.4 of [13], we can assume that (f, g) has $N(f, g; X, A)$ coincidence points. By Theorem 3.23, we can move any coincidence point $x \in X - A$ in a weakly common coincidence class to A. $\square$

4. Surplus coincidence theory

Our aim in this section is to give a sharp lower bound for $M(f, g; X - A)$ when the subspace does not satisfy the by-passing condition. Let $f, g : (X, A) \to (Y, B)$ and $\hat{A} = \bigcup_1^l A_k$ be as in Section 3. We will be considering coincidence point classes on the complement, i.e., $V = X - A$ in the definitions of Section 2. By abuse we write $[x]$ instead of $[x]_{X-A}$. Since all classes considered in this section are on $X - A$ this should cause no confusion.

Note the similarity of the next definition with the conclusion of Lemma 3.20. Note also however that the by-passing condition is absent.

Definition 4.1. A coincidence point x on $X - A$ is said to be nonsurplus if there is a path $\alpha : (I, 0, I - \{1\}, 1) \to (X, x, X - A, A)$ from x to A such that $f(\alpha) \simeq g(\alpha) : (I, 0, 1) \to (Y, f(x), B)$. A coincidence point of f and g on $X - A$ that is not a nonsurplus coincidence point is said to be surplus.

It is trivial to show that if $[y]$ is a coincidence point class of f and g on $X - A$, and if $x \in [y]$, is surplus (nonsurplus) then so also is every $x \in [y]$.

Proposition 4.2. *The coincidence point classes of f and g on $X - A$ divide the coincidence points on $X - A$ into surplus and nonsurplus classes.*

The proof of the following theorem is similar to the proof of [18, Theorem 3.3].

Theorem 4.3. *The number of surplus coincidence classes of f and g on $X - A$ is finite, and each of them is a compact subset of X.*

From the comments in Section 2 we can now see that each surplus coincidence class has an index.

Definition 4.4. A surplus coincidence class of f and g on $X - A$ is said to be essential if it has nonzero index. The number of essential surplus coincidence classes is called the surplus coincidence Nielsen number of f and g on $X - A$, and is denoted by $SN(f, g; X - A)$.

We need the following lemma for homotopy invariance.

Lemma 4.5. *If $F : f \simeq f_1$ and $G : g \simeq g_1$ are homotopies and the classes $[x]$ and $[y]$ are F,G related, then $[x]$ is a surplus coincidence point class if and only if $[y]$ is.*

Proof. We sketch the proof which is similar to the proof of [19, Lemma 3.5]. We use the contrapositive. Suppose that $[x]$ and $[y]$ are F,G related and $[x]$ is nonsurplus. Then $[x]$ and $[y]$ are, respectively the zero and 1 slices of a single coincidence point class of the fat homotopies $\boldsymbol{F}, \boldsymbol{G} : X \times I \to Y \times I$ on the complement $(X \times I) - (A \times I)$. Let $x \in [x]$ then it is easy to see that $(x, 0)$ is a nonsurplus point of $\boldsymbol{F}$ and $\boldsymbol{G}$ on the complement $(X \times I) - (A \times I)$. Thus by the remarks preceding Proposition 4.2 every coincidence point of the class containing $(x, 0)$ is nonsurplus. In particular, $(y, 1)$, and hence y (use the projection $Y \times I \to Y$) is nonsurplus for any $y \in [y]$. $\square$

The next theorem now follows from Theorem 2.1 and from Lemma 4.5.

Theorem 4.6. *$SN(f, g; X - A)$ is a homotopy invariant.*

Theorem 4.7. *For any pair of maps* $f, g : (X, A) \to (Y, B)$ *there are at least* $SN(f, g; X - A)$ *coincidence points on* $X - A$.

Proof. Clearly each essential surplus coincidence class contains a coincidence point on $X - A$, the result follows. □

The proof of the next theorem is similar to the proof of [19, 3.8].

Theorem 4.8. *Let* $f, g : (X, A) \to (Y, B)$ *be maps, then* $SN(f, g; X - A) \geqslant N(f, g; X - A)$. *Furthermore if* A *can be by-passed in* X, *then* $SN(f, g; X - A) = N(f, g; X - A)$.

Example 4.9. In Example 1.1, since the two surplus points $e^{2\pi i/3}$ and $e^{4\pi i/3}$ are in separate components of $X - A$, they determine distinct surplus classes. These classes are essential, and so $SN(f, g; X - A) = 2$. Thus we see that the two coincidence points in the complement in this example cannot be moved to A by relative homotopies of f and g.

4.2. The minimum theorem, closing remarks

In this last subsection of the paper we prove the minimum theorem in the context of surplus theory, and make some closing remarks. The first thing to observe is that we have organized the paper so that the proofs of Theorems 3.23 and 4.10 are as close as possible to being identical. In fact much of the preparation for Theorem 4.10 comes in Section 3. The difference in the two proofs comes in the reason for the existence of the paths that make the various parts work. In Theorem 3.23 the the appropriate paths exist since the subspaces involved satisfy the by-passing condition. In Theorem 4.10 the existence of the appropriate paths comes directly from the definitions of surplus theory.

Theorem 4.10. *Let* $f, g : (X, A) \to (Y, B)$ *be maps of a pairs of manifolds with* $A \subseteq X$, $B \subseteq Y$ *locally flat. If* $\dim(X) \geqslant 3$, *then there is a map* $f' \simeq f : (X, A) \to (Y, B)$ *such that* (f', g) *has* $SN(f, g; X - A)$ *coincidence points on* $X - A$.

Proof. As in Theorem 3.23 we may assume that there are only finite number of coincidence points on X. Now if x and y lie in the same surplus fixed point class on $X - A$ then by definition x and y may be joined by a path α in $X - A$ with $f(\alpha) \sim g(\alpha)$. So again as in the proof of Theorem 3.23 any two such x and y may be coalesced, and those with zero index eliminated. We may therefore assume that each of the remaining $SN(f, g; X - A)$ surplus classes on $X - A$ contains exactly one coincidence point. The nonsurplus coincidence points on $X - A$ can now be moved to A by Lemma 3.22. □

Remark 4.11. There is an analogue of Theorem 3.24 for this section too, but it requires that a new Nielsen type number $SN(f, g; X, A)$ to be defined as follows

$$\begin{aligned} SN(f, g; X, A) := {} & N(f, g; X, A) + SN(f, g; X - A) - N(f, g; X - A) \\ = {} & N(f_A, g_A) + SN(f, g; X - A) + E(f, g; f_A, g_A) \\ & - N(f, g; f_A, g_A). \end{aligned}$$

This number satisfies the same properties as the surplus Nielsen numbers, i.e., homotopy invariance, it is a lower bound for the number of coincidences on X. For Example 1.1 $NS(f, g; X, A) = 4$, and finally under the same hypotheses as Theorem 3.24, there is a minimum theorem.

Clearly all this specializes to the fixed point case by Theorem 2.14 (this specialization is also new).

References

[1] B. Brooks, Coincidences, roots, and fixed points, Thesis, University of California, Los Angeles (1967).
[2] B. Brooks, On removing coincidences of two maps when only one, rather than both, of them may be deformed by a homotopy, Pacific J. Math. 40 (1972) 45–52.
[3] R. Brown, Lefschetz Fixed Point Theorem (Scott, Freshman and Company, 1971).
[4] R. Brown and H. Schirmer, Nielsen coincidence theory and coincidence-producing maps for manifolds with boundary, Topology Appl. 46 (1992) 65–79.
[5] J. Guo, Relative and equivariant coincidence theory, Ph.D. Thesis, Memorial University of Newfoundland (1996).
[6] J. Guo, On a relative theorem of Brooks, In preparation.
[7] J. Guo and P.R. Heath, Equivariant coincidence Nielsen numbers (announcement), Preprint.
[8] J. Guo and P.R. Heath, Equivariant coincidence Nielsen numbers (full version), In preparation.
[9] P.R. Heath, Product formulae for Nielsen numbers of fibre maps, Pacific J. Math. 117 (1985) 267–289.
[10] P.R. Heath, R. Piccinini and C. You, Nielsen-type numbers for periodic points, I, in: Topological Fixed Point Theory and Applications, Proceedings, Tianjin 1988, Lecture Notes in Math. 1411 (Springer, Berlin, 1989) 88–106.
[11] B. Jiang, Lectures on Nielsen Fixed Point Theory, Comtemporary Mathematics 14 (Amer. Math. Soc., Providence, RI, 1983).
[12] C.G. Jang and S. Lee, A relative Nielsen number in coincidence theory, J. Korean Math. Soc. 32 (1995) 171–181.
[13] J. Jezierski, The relative coincidence Nielsen number, Fund. Math. 149 (1996) 1–18.
[14] S. Lee, A relative Nielsen coincidence number for the complement I, J. Korean Math. Soc. 33 (1996) 709–716.
[15] H. Schirmer, Mindestzahlen von Koinzidenzpunkten, J. Reine Angew. Math. 194 (1955) 21–39.
[16] H. Schirmer, A relative Nielsen number, Pacific J. Math. 122 (1986) 459–473.
[17] J. Vick, Homology Theory (Academic Press, Inc., New York, 1973).
[18] X. Zhao, A relative Nielsen number for the complement, in: Topological Fixed Point Theory and Applications, Lecture Notes in Math. 1411 (Springer, Berlin, 1989) 189–199.
[19] X. Zhao, Estimation of the number of fixed points on the complement, Topology Appl. 37 (1990) 257–265.